

# On the symmetric formulation of the Painlevé IV equation

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## ABSTRACT

Symmetries and solutions of the Painlevé IV equation are presented in an alternative framework which provides the bridge between the Hamiltonian formalism and the symmetric Painlevé IV equation. This approach originates from a method developed in the setting of pseudo-differential Lax formalism describing AKNS hierarchy with the Darboux-Bäcklund and Miura transformations.

In the Hamiltonian formalism the Darboux-Bäcklund transformations are introduced as maps between solutions of the Hamilton equations corresponding to two allowed values of Hamiltonian's discrete parameter. The action of the generators of the extended affine Weyl group of the  $A_2$  root system is realized in terms of three “square-roots” of such Darboux-Bäcklund transformations defined on a multiplet of solutions of the Hamilton equations.

## 1 Introduction

The six Painlevé equations arise as special scaling limits of integrable models and frequently emerge in various discrete and continuous models of physics, many-body systems and mathematics. They were originally proposed by Painlevé as the second-order differential equations whose solutions have no movable singular points except poles. The Painlevé equations occupy a central place in the study of nonlinear systems in view of their applications to a variety of cross-disciplinary problems. Consequently there exists a vast body of literature devoted to a remarkable connection between these equations and wide-range of problems of mathematical physics. In the past a lot of effort has been invested in finding all rational solutions and their symmetry structure (see f.i. [18, 12, 6, 7] for the case of the Painlevé IV equation).

Our work is devoted towards constructing universal approach to deal with symmetries and solutions of the Painlevé equations originating from methods we have previously introduced for a class of integrable models described by the pseudo-differential Lax operators encountered in the Kadomtsev-Petviashvili type of hierarchies. Recently, in [2], we reported progress in utilizing techniques of integrable models for the purpose of obtaining and classifying solutions of the Painlevé IV equations. In particular, reference [2] explored the connection between the pseudo-differential Lax hierarchies describing various versions of the AKNS hierarchy subject to the string equation and the Painlevé IV equation for the purpose of reproducing in a novel way the rational solutions of the Painlevé IV equation through the Darboux-Bäcklund (DB) transformations of the reduced integrable hierarchy.

In this paper we use a generalization of Okamoto's Hamiltonian system [19], which explicitly depends on four parameters. Three of the parameters,  $(v_i, v_j, v_k)$ , labeled by distinct integers  $i, j, k = 1, 2, 3$ , satisfy the condition  $\sum_n v_n = 0$  and are associated with the root system of the  $A_2$  Lie algebra. A special role is played by,  $\epsilon$ , the additional fourth parameter in the Hamiltonian (3.1). For two values of  $\epsilon$ ,  $+1$  and  $-1$ , the Hamilton equations reproduce the Painlevé IV equation:

$$y_{xx} = \frac{1}{2y}y_x^2 + \frac{3}{2}y^3 + 4xy^2 + 2(x^2 + b)y - 2\frac{a}{y} \quad (1.1)$$

where  $a$  and  $b$  are parameters. When  $\epsilon$  changes its value from e.g  $+1$  to  $-1$  the parameter  $b$  shifts by 2. In the Hamiltonian setting the Darboux-Bäcklund transformation is defined as a map between two solutions of the Hamilton equations corresponding to the two allowed values of the “epsilon” parameter. In case of the Painlevé IV equation such transformations agree with the Darboux-Bäcklund transformations of an underlying Lax operator of the reduced AKNS hierarchy. In addition to the Darboux-Bäcklund transformations we define simple permutation operations which preserve the form of the Hamiltonian but permute the  $(v_i, v_j, v_k)$  parameters and the corresponding solutions of the Hamilton equations. Defining Hamiltonian for the Painlevé IV equation in the form which is left invariant under the permutation operators, allows us to study their group symmetries in a natural manner. The study leads to realization of the Darboux-Bäcklund transformations  $G_n, n = 1, 2, 3$  as shift operators acting on the parameters according to :

$$G_n(v_l) = v_l + \frac{1}{3} - \delta_{n,l}, \quad n, l = 1, 2, 3. \quad (1.2)$$

The action of the extended affine Weyl group generators associated with the roots is realized in terms of the “square-roots”  $g_n, n = 1, 2, 3$  of the Darboux-Bäcklund transformations such that  $G_n = g_n^2$  for  $g_n$  defined on a multiplet of solutions of the symmetric Hamilton equations and satisfying algebraic relations :

$$g_n g_m g_n = g_m g_n g_m, \quad (g_n g_m g_n)^2 = 1, \quad (g_n^2 g_m)^2 = 1, \quad (g_m g_n^2)^2 = 1, \quad (1.3)$$

for distinct  $n, m = 1, 2, 3$ .

The results of this study provide insight into the structure of solutions and connections between different symmetric realizations of the Painlevé equations. There

are several advantages of the approach we are proposing. First, it reproduces the symmetric form of the Painlevé IV system invariant under extended affine Weyl group from a single scalar equation for the Jimbo-Miwa-Okamoto  $\rho$  function [11, 19] and its simple symmetry structure. Secondly, the underlying Darboux dressing chain described in relations (3.22) and (3.58) allows for establishing simple explicit relations between elements of the symmetric Painlevé equation and the underlying Jimbo-Miwa-Okamoto  $\rho$ -function. This equivalence reduces the task of finding rational solutions of the Painlevé IV equation to finding polynomial solutions of the  $\rho$ -function equation and acting on them with Bäcklund transformations realized as compositions of permutation operations and Darboux-Bäcklund transformations.

The paper is organized as follows. In Section 2, we briefly review the constrained AKNS hierarchy emphasizing its symmetry under Darboux-Bäcklund transformations operating on various pseudo-differential Lax representations of the hierarchy. We also derive an equation for the Jimbo-Miwa-Okamoto function  $\rho = (\ln \tau)_x$ , for the tau-function  $\tau$ , which is fully equivalent to the Painlevé IV equation. In Section 3, the Hamiltonian approach is shown to be equivalent with the symmetric Painlevé IV formalism. The “square-roots” of the Darboux-Bäcklund transformations are shown to give rise to realization of the affine Weyl group symmetry of  $A_2$ . It is also shown how the equation for  $\rho$  follows from the symmetric Painlevé IV equations. Although material in Section 3 is inspired by Section 2 both sections can be read independently. Finally, in Section 4 the symmetry of the  $\rho$  equation is applied towards construction of rational solutions of the Painlevé IV equation.

## 2 AKNS hierarchy, string condition and Bäcklund transformations

We show in this section how the pseudo-differential Lax formalism of the AKNS hierarchy with the Darboux-Bäcklund symmetry augmented by the Virasoro constraint can be used to uncover the symmetry structure of the Painlevé IV equation. Deriving the Painlevé equations as limits of integrable soliton equations has an advantage that the symmetry transformations which are present in the original integrable model and which survive the imposed limit can effectively be used to produce all rational solutions of the Painlevé equations out of few basic polynomial solutions. Recent insight into close links of the Painlevé equations with the tau function approach [3] and even more fundamentally with the Lax formalism has opened new ways to tackle the problem of uncovering a rich symmetry structure among solutions but also provided a direct method to generate solutions of the Painlevé equations. In particular, the Darboux-Bäcklund transformations of integrable models provide a natural framework for the derivation of symmetry of the Painlevé IV equation [2].

This section is based on two observations put forward in [2]. First, the Painlevé IV equation is obtained as reduction of the AKNS hierarchy subject to the additional non-isospectral Virasoro symmetry constraint (also known as the string equation). Next, we recall from [4] that the Darboux-Bäcklund transformations commute with the additional-symmetry Virasoro flows and thus for that reason they induce

symmetry operations on the Painlevé IV equation. The Miura map between two different Lax realizations of the AKNS hierarchy [1] is then used to reveal a more detailed structure of solutions and their symmetries. This structure will give rise to an explicit construction of the symmetric Painlevé IV equation.

## 2.1 Reduced AKNS Lax formalism

### 2.1.1 String equation

To introduce the AKNS Lax formalism with the additional Virasoro symmetry flows we define the pseudo-differential Lax operator :

$$L = \partial_x - r\partial_x^{-1}q. \quad (2.1)$$

The associated isospectral  $t_n$ -flow defined through the Lax equation:  $\partial L / \partial t_n = [L_+^n, L]$  amounts for  $n = 2$  to :

$$\frac{\partial}{\partial t_2}q + q_{xx} - 2q^2r = 0, \quad \frac{\partial}{\partial t_2}r - r_{xx} + 2qr^2 = 0, \quad \frac{\partial}{\partial t_2}\rho = 2(-qr_x + rq_x). \quad (2.2)$$

The first two equations reproduce the conventional AKNS equations for the potentials  $r$  and  $q$ . The third equation describes a flow of the so-called squared eigenfunction  $\rho$  (see f.i. [5]), such that  $\rho_x = -2rq$ .

The AKNS hierarchy can be augmented by infinitely many Virasoro symmetry flows [20], which commute with the isospectral flows. A closed subset of three additional Virasoro flows, which forms the  $sl(2)$  subalgebra of the Virasoro algebra, has been shown to preserve the form of the Lax operator of the AKNS hierarchy [4]. As in [2], we reduce the original AKNS hierarchy by setting one of these flows to zero. That amounts to imposing the following “string equation”:

$$-xq_x - 2t_2 \frac{\partial}{\partial t_2}q = q + \nu q, \quad xr_x + 2t_2 \frac{\partial}{\partial t_2}r = -r + \nu r, \quad x\rho_x + 2t_2 \frac{\partial}{\partial t_2}\rho = -\rho, \quad (2.3)$$

where  $\nu$  is a free parameter. One further reduces the hierarchy by setting  $t_3$  and all the higher flows to zero and using the AKNS equation (2.2) to eliminate the  $t_2$ -dependence from  $r, q, \rho$  for the fixed  $t_2 = -1/4$  value. This procedure turns equation (2.3) into the constraint given by

$$\begin{aligned} -xq_x + \frac{1}{2}(-q_{xx} + 2q^2r) &= q + \nu q, & xr_x - \frac{1}{2}(r_{xx} - 2qr^2) &= -r + \nu r \\ \rho + x\rho_x &= \rho - 2xrq = q_xr - qr_x \end{aligned} \quad (2.4)$$

Few simple algebraic steps followed by an integration (see [2]) yield from eq. (2.4) :

$$q_xr_x = q^2r^2 + 2x\rho - 2\nu r q - (\mu^2 - \nu^2), \quad (2.5)$$

with  $\mu^2 - \nu^2$  emerging as an integration constant with a new parameter  $\mu$ . In what follows  $\mu$  together with  $\nu$  will parametrize solutions of the Painlevé IV equation.

Dividing the first of eqs. (2.4) by  $q$  and second by  $r$  and summing them and multiplying the result by  $rq$  produces an equation :

$$-x^2\rho_x + x\rho + \frac{1}{4}\rho_{xxx} + 2\nu\rho_x + \frac{3}{4}\rho_x^2 = \mu^2 - \nu^2, \quad (2.6)$$

entirely expressed in terms of only one variable  $\rho$ . Equation (2.6) was obtained after inserting  $q_x r_x$  from eq. (2.5) and eliminating the product of  $r$  and  $q$  through  $rq = -\rho_x/2$ .

Equation (2.6) is a special case of the Chazy I equation [8]. It can be integrated into [11] :

$$\rho_{xx}^2 = 4(x\rho_x - \rho)^2 - 2\rho_x^3 - 8\nu\rho_x^2 + 8(\mu^2 - \nu^2)\rho_x - 8C, \quad (2.7)$$

with  $C$  being another integration constant. Setting the integration constant  $C$  to zero simplifies eq. (2.7) to :

$$(2(x\rho_x - \rho) + \rho_{xx})(2(x\rho_x - \rho) - \rho_{xx}) = 2\rho_x[\rho_x - 2(\mu - \nu)][\rho_x + 2(\mu + \nu)]. \quad (2.8)$$

It is well-known (see f.i. [6, 2]) that two functions defined as

$$\begin{aligned} y_+ &= -\frac{q_x}{q} - 2x = \frac{-1}{2\rho_x}(2(x\rho_x - \rho) + \rho_{xx}) \\ y_- &= \frac{r_x}{r} - 2x = \frac{-1}{2\rho_x}(2(x\rho_x - \rho) - \rho_{xx}) \end{aligned} \quad (2.9)$$

satisfy the Painlevé IV equation (1.1) with  $b = \nu \pm 1$  and  $a = \mu^2$  for  $y = y_{\pm}$ .

## 2.2 The Darboux-Bäcklund transformations

In the context of the AKNS Lax hierarchy we consider the Darboux-Bäcklund transformation realized as a similarity transformation by an operator  $T = r\partial_x r^{-1}$  :

$$L = \partial_x - r\partial_x^{-1}q \rightarrow \bar{L} = T L T^{-1} = \partial_x - \bar{r}\partial_x^{-1}\bar{q}.$$

This transformation is known to leave the AKNS Lax equations invariant. It is also known that the DB transformations commute with the additional-symmetry Virasoro flows [4] and consequently these transformations will also leave equation (2.8) invariant since it was obtained as a reduction of the AKNS equation under an additional Virasoro constraint. Here, we will show how to use the DB transformations to transform solutions of equation (2.8) to other solutions of these equations characterized by new values of the parameter  $\nu$ .

A simple calculation yields

$$\bar{r} = r(\ln r)_{xx} - r^2 q, \quad \bar{q} = -\frac{1}{r}. \quad (2.10)$$

It is equally easy to formulate the adjoint Darboux-Bäcklund transformation, generated by acting with  $S^* = q^{-1}\partial_x q$  on the pseudo-differential Lax operator through the following similarity transformation :

$$L = \partial_x - r\partial_x^{-1}q \rightarrow \tilde{L} = S^{*-1} L S^* = (q^{-1}\partial_x^{-1}q)L(q^{-1}\partial_x q) = \partial_x - \tilde{r}\partial_x^{-1}\tilde{q}$$

with

$$\tilde{q} = -q(\ln q)_{xx} + q^2 r, \quad \tilde{r} = \frac{1}{q}. \quad (2.11)$$

It is convenient to rewrite actions of  $\sim$  and  $-$  in eqs. (2.11) and (2.10) as, respectively, transformations  $G$  and  $G^{-1}$  acting on variables  $J, \bar{J}$  defined as :

$$\bar{J} = -rq = \rho_x/2, \quad J = (\ln q)_x = -y_+ - 2x.$$

In terms of the above variables the DB transformations from eqs. (2.11) and (2.10) take the following form [1] :

$$G(J) \equiv J + (\ln(\bar{J} + J_x))_x \quad G(\bar{J}) \equiv \bar{J} + J_x \quad (2.12)$$

$$G^{-1}(J) \equiv J - (\ln \bar{J})_x \quad G^{-1}(\bar{J}) \equiv \bar{J} + (\ln \bar{J})_{xx} - J_x. \quad (2.13)$$

It follows from (2.12)-(2.13) and relation (2.9) that:

$$G(y) = y - (\ln(y_x + y^2 + 2xy + 2\nu + 4))_x, \quad G^{-1}(y) = y + (\ln(y_x - y^2 - 2xy - 2\nu))_x, \quad (2.14)$$

where for notational simplicity we set  $y$  for  $y_+$  from relation (2.9). The DB transformations  $G^{\pm 1}$  agree with Murata's transformations  $T_{\mp}$  [15] after identifying parameters  $\theta, \alpha$  from [15] with  $\mu, -\nu - 1$ . The action of  $G^{\pm 1}$  maps the parameters  $\mu, \nu$  to  $\mu, \nu \pm 2$ .

There also exists a set of variables  $j, \bar{j}$  entering a different alternative pseudo-differential Lax realization of the AKNS hierarchy. These variables are related to  $J, \bar{J}$  via a Miura transformation [1] :

$$J = -j - \bar{j} + \frac{j_x}{j}; \quad \bar{J} = \bar{j} j. \quad (2.15)$$

In terms of variables  $j, \bar{j}$  one can define a “square-root” of  $G$  transformation as :

$$g(j) \equiv \bar{j} - \frac{j_x}{j} \quad g(\bar{j}) \equiv j \quad (2.16)$$

$$g^{-1}(\bar{j}) \equiv j + \frac{\bar{j}_x}{\bar{j}} \quad g^{-1}(j) \equiv \bar{j}, \quad (2.17)$$

such that the following relation [1] :

$$g^2 = G, \quad (2.18)$$

holds when both sides are applied on  $J, \bar{J}$  defined by equation (2.15).

From the above relations one obtains simple transformation rules:

$$g(y) = y - (\ln(-j + y + 2x))_x, \quad g^{-1}(y) = y + (\ln(j))_x, \quad (2.19)$$

which agree in general with transformations derived using the Schlesinger equations in [9]. In the above equation again for notational simplicity  $y$  denotes  $y_+$  from relation (2.9).

The Bäcklund transformation  $g$  when applied on solutions expressed by  $j, \bar{j}$  shifts both  $\nu$  and  $\mu$  by 1 (see subsection 3.1 for details in terms of  $v_n$  parameters) in such a way that acting twice with  $g$  agrees with the formula (2.18). Because of the property of  $g$  transformation to shift both parameters of the Painlevé IV equation this transformation is very useful in deriving solutions corresponding to new values of the parameters [2].

It turns out that both  $-j$  and  $-\bar{j}$ , in addition to  $y_+$ , are solutions to the Painlevé IV equation. In the next section we will present a systematic approach to dealing with a presence of all the solutions  $y_\pm, -j$  and  $-\bar{j}$  which emerged here from the AKNS hierarchy structure behind the Painlevé IV equation.

### 3 Symmetry of the Painlevé IV equation

#### 3.1 Hamiltonian approach to the Painlevé IV equation

The formalism is defined in terms of the following generalization of the Painlevé IV Hamiltonian from [19] :

$$H = 2P^2Q - \epsilon (Q^2 + 2xQ + 2(v_j - v_i)) P + (v_k - v_i)Q - 2v_i x, \quad (3.1)$$

where  $\epsilon$  is a constant and the parameters  $v_n, n = 1, 2, 3$  satisfy condition  $\sum_n v_n = 0$  and where  $i, j, k$  are fixed distinct numbers between 1 and 3. The resulting Hamilton equations are:

$$Q_x = \frac{\partial H}{\partial P} = 4QP - \epsilon (Q^2 + 2xQ + 2(v_j - v_i)) \quad (3.2)$$

$$P_x = -\frac{\partial H}{\partial Q} = -2P^2 + \epsilon (2QP + 2xP) - (v_k - v_i). \quad (3.3)$$

For two values of  $\epsilon$  such that  $\epsilon^2 = 1$  the equation for  $Q_{xx}$  derived from (3.2)-(3.3) (after elimination of  $P$ ) can be cast in the form of the Painlevé IV equation (1.1) with

$$a = (v_j - v_i)^2, \quad b = -\epsilon - 3v_k. \quad (3.4)$$

Due to the Hamilton equations it holds that

$$\frac{d}{dx}H = H_x = -2\epsilon QP - 2v_i. \quad (3.5)$$

By taking a derivative of (3.5), one obtains

$$H_{xx} = \epsilon Q (H_x + 2v_k) + 2P (H_x + 2v_j).$$

Furthermore by combining relation (3.5) with the definition of  $H$  one finds that

$$2(xH_x - H) = -Q (H_x + 2v_k) + 2\epsilon P (H_x + 2v_j). \quad (3.6)$$

One now easily derives expressions for  $Q$  and  $P$  in terms of the Hamiltonian  $H$  and its derivatives :

$$Q^{(k)} = \frac{2(xH_x - H) - \epsilon H_{xx}}{(-2)(H_x + 2v_k)}, \quad 2P^{(j)} = \frac{2(xH_x - H) + \epsilon H_{xx}}{(2\epsilon)(H_x + 2v_j)}, \quad (3.7)$$

where we labeled  $Q$  and  $P$  by an index of the  $v$  parameter appearing in their denominators.

Plugging the values of  $Q$  and  $P$  from expressions (3.7) into the relation  $2QP = 2Q^{(k)}P^{(j)} = -\epsilon(H_x + 2v_i)$  yields

$$(2(xH_x - H) - H_{xx})(2(xH_x - H) + H_{xx}) = 4 \prod_{n=1}^3 (H_x + 2v_n), \quad (3.8)$$

Let us define an operation  $\pi_{i,k}$  exchanging parameters  $v_i$  and  $v_k$  into each other according to :

$$\pi_{i,k}(v_i, v_j, v_k) = (v_k, v_j, v_i) \quad (3.9)$$

for distinct  $i, j, k$ . Equation (3.8) is manifestly invariant under all three permutations  $\pi_{i,j}$  for  $\pi_{i,j}(H) = H$ .

The permutation  $\pi_{i,k}$  transforms  $Q^{(k)}$  in expression (3.7) to  $Q^{(i)}$  by replacing  $v_k$  with  $v_i$ . Accordingly,  $\pi_{i,k}$  changes of the parameters  $a, b$  (3.4) in the Painlevé IV equation (1.1) from  $a_k = (v_j - v_i)^2, b_k = -\epsilon - 3v_k$  to  $a_i = (v_k - v_j)^2, b_i = -\epsilon - 3v_i$  or

$$b_i = -\frac{3}{2}\epsilon - \frac{1}{2}b_k + \frac{3}{2}\eta\sqrt{a_k}, \quad a_i = (b_k + \epsilon + \eta\sqrt{a_k})^2/4, \quad \eta = \pm 1$$

in the form in which these types of the Bäcklund transformations first appeared in [14, 13].

Define function  $\rho^{(k)}$  such that

$$\rho_x^{(k)} = 2H_x + 4v_k. \quad (3.10)$$

Then plugging  $\rho = \rho^{(k)}$  into equation (3.8) reproduces eq. (2.8), with  $\mu$  and  $\nu$  given by  $\mu^2 = (v_j - v_i)^2$  and  $\nu = -3v_k$ . Therefore by comparing with eq. (3.4) and expression for  $Q$  in relation (3.7) we find that

$$y_{\pm} = Q_{-\epsilon}^{(k)} = \frac{-1}{2\rho_x} (2(x\rho_x - \rho) \pm \rho_{xx}) = \frac{-1}{2\rho_x} (2(x\rho_x - \rho) - \epsilon\rho_{xx}), \quad \epsilon = \mp 1 \quad (3.11)$$

will solve the Painlevé IV equation (1.1) with  $a = \mu^2, b = \nu \pm 1$ .

Next, we will study symmetry of equation (2.8) for the  $\rho$ -function. One notes that the left hand side of equation (2.8) remains invariant under substitution  $\rho = \tilde{\rho} + Cx$  for any constant  $C$ . However for two values of  $C$ , namely  $C = 2(\mu - \nu)$  and  $C = -2(\mu + \nu)$ , the right hand side can be given the form  $2\tilde{\rho}_x[\tilde{\rho}_x - 2(\tilde{\mu} - \tilde{\nu})][\tilde{\rho}_x + 2(\tilde{\mu} + \tilde{\nu})]$  with

$$\tilde{\rho} = \rho^{(i)} = \rho - 2(\mu - \nu)x, \quad \nu^{(i)} = \frac{3}{2}\mu - \frac{1}{2}\nu, \quad \mu^{(i)} = \pm \frac{1}{2}(\mu + \nu) \quad (3.12)$$

$$\tilde{\rho} = \rho^{(j)} = \rho + 2(\mu + \nu)x, \quad \nu^{(j)} = -\frac{3}{2}\mu - \frac{1}{2}\nu, \quad \mu^{(j)} = \pm \frac{1}{2}(\mu - \nu). \quad (3.13)$$

Thus for the above two values of constant  $C$  the transformation  $\rho \rightarrow \rho^{(n)}$  takes the “old” solution of equation (2.8) to the “new” solution of equation (2.8) with the new parameters  $\mu^{(n)}, \nu^{(n)}$  for  $n = i, j$ . Note, that (3.13) is obtained from (3.12) by  $\mu \rightarrow -\mu$ . Since  $\rho$  is a solution of equation (2.8), which only depends on a parameter  $\mu^2$  it is therefore invariant under  $\mu \rightarrow -\mu$ . One sees by inspection that eq. (2.8) can be rewritten as

$$\begin{aligned} (2(x\rho_x^{(n)} - \rho^{(n)}) + \rho_{xx}^{(n)})(2(x\rho_x^{(n)} - \rho^{(n)}) - \rho_{xx}^{(n)}) &= 2\rho^{(i)}\rho^{(j)}\rho^{(k)} \\ &= 2\rho_x^{(n)}[\rho_x^{(n)} - 2(\mu^{(n)} - \nu^{(n)})][\rho_x^{(n)} + 2(\mu^{(n)} + \nu^{(n)})], \quad n = i, j, k, \end{aligned} \quad (3.14)$$

where we used notation  $\rho^{(k)} = \rho, \nu^{(k)} = \nu, \mu^{(k)} = \mu$ . Also, note that  $\sum_{i=1}^3 \nu^{(i)} = 0$ .

One verifies that  $\rho_x^{(i)} = 2H_x + 4v_i$  and  $\rho_x^{(j)} = 2H_x + 4v_j$ . Recall that  $\rho_x = 2H_x + 4v_k = \rho_x^{(k)}$ . On basis of the definition (3.9) it follows that

$$\pi_{k,j}(\mu, \nu) = (\mu^{(j)}, \nu^{(j)}), \quad \pi_{k,i}(\mu, \nu) = (\mu^{(i)}, \nu^{(i)}) \quad (3.15)$$

It also follows that  $\pi_{j,i}(\mu^{(j)}, \nu^{(j)}) = (\mu^{(i)}, \nu^{(i)})$  and in agreement with the definition (3.9) that :

$$\pi_{k,i}(\rho^{(k)}) = \rho^{(i)}, \quad \pi_{k,j}(\rho^{(k)}) = \rho^{(j)}. \quad (3.16)$$

In accordance with (3.12) and (3.13) we now extend definition (3.11) of solutions to the Painlevé IV equation to :

$$y_{\pm}^{(n)} = \frac{-1}{2\rho_x^{(n)}} (2(x\rho_x^{(n)} - \rho^{(n)}) \pm \rho_{xx}^{(n)}), \quad n = 1, 2, 3, \quad (3.17)$$

with  $y_{\pm}^{(k)} = y_{\pm}$ . Substituting in the above definition the expressions of  $\rho^{(n)}$  yields

$$y_{\pm}^{(i)} = Q_{-\epsilon}^{(i)} = \frac{-1}{2(\rho_x - 2(\mu - \nu))} (2(x\rho_x - \rho) \pm \rho_{xx}) \quad (3.18)$$

$$y_{\pm}^{(j)} = Q_{-\epsilon}^{(j)} = \frac{-1}{2(\rho_x + 2(\mu + \nu))} (2(x\rho_x - \rho) \pm \rho_{xx}) \quad (3.19)$$

Note that  $y_+^{(i)}$  solves the Painlevé IV equation with  $\mu^{(i)}, \nu^{(i)}$  from (3.12) and  $y_-^{(i)}$  solves the Painlevé IV equation with  $\mu^{(i)}, \nu^{(i)} - 2$ . Similarly,  $y_+^{(j)}$  solves the Painlevé IV equation with  $\mu^{(j)}, \nu^{(j)}$  from (3.13) and  $y_-^{(j)}$  solves the Painlevé IV equation with  $\mu^{(j)}, \nu^{(j)} - 2$ . Comparing with equation (3.7) one sees that  $-2\epsilon P^{(i)} = y_{\mp}^{(i)}, -2\epsilon P^{(j)} = y_{\mp}^{(j)}$  for the two allowed values of  $\epsilon$ . They reproduce functions  $-j$  and  $-\bar{j}$  found in subsection 2.2 within the AKNS hierarchy.

Based on relation (3.16) the symmetry operations defined in (3.9) transform different solutions  $y_{\pm}^{(i)}$  into each other according to

$$\pi_{i,k}(y_{\pm}^{(i)}) = y_{\pm}^{(k)}. \quad (3.20)$$

### 3.2 Symmetric Painlevé IV equations

It is a simple consequence of equation (2.8) that

$$y_{\pm}^{(i)} y_{\mp}^{(j)} = \frac{1}{2} \rho_x^{(k)}, \quad i, j, k = 1, 2, 3, \quad (3.21)$$

for distinct  $i, j, k$ . This equation is manifestly invariant under all three  $\pi_{i,j}$  transformations.

Inserting expressions (3.5) and (3.10) into the Hamiltonian equation (3.2) yields

$$\rho_x = -\epsilon Q_x - Q^2 - 2xQ - 2(-3v_k),$$

which after recalling that  $\epsilon = \pm 1$  corresponds to  $y_{\mp}^{(n)}$  can be rewritten as :

$$\rho_x^{(n)} = y_{+x}^{(n)} - (y_+^{(n)})^2 - 2xy_+^{(n)} - 2\nu^{(n)} = -y_{-x}^{(n)} - (y_-^{(n)})^2 - 2xy_-^{(n)} - 2\nu^{(n)}, \quad (3.22)$$

for  $n = 1, 2, 3$ .

Due to an elementary relation

$$\rho_x^{(j)} + 2\nu^{(j)} = \frac{1}{2} \rho_x^{(i)} + \frac{1}{2} \rho_x^{(k)}, \quad (3.23)$$

valid for distinct  $i, j, k$ , we get from equations (3.21) and (3.22) the following two identities:

$$y_+^{(j)} y_-^{(k)} + y_+^{(j)} y_-^{(i)} = y_{+x}^{(j)} - (y_+^{(j)})^2 - 2xy_+^{(j)} \quad (3.24)$$

$$y_-^{(j)} y_+^{(k)} + y_-^{(j)} y_+^{(i)} = -y_{-x}^{(j)} - (y_-^{(j)})^2 - 2xy_-^{(j)} \quad (3.25)$$

We can summarize the above equations as :

$$y_{\mp}^{(i,j)} + y_{\mp}^{(j)} + y_{\pm}^{(k)} = -2x, \quad (3.26)$$

for distinct  $i, j, k$  and with

$$y_{\mp}^{(i,j)} = y_{\pm}^{(i)} \pm \left( \ln \left( y_{\mp}^{(j)} \right) \right)_x.$$

Consider one of equations in (3.26), e.g.  $y_-^{(i,j)} + y_-^{(j)} + y_+^{(k)} = -2x$  together with

$$y_+^{(i)} y_-^{(j)} = \left( y_-^{(i,j)} - \left( \ln \left( y_-^{(j)} \right) \right)_x \right) y_-^{(j)} = \frac{1}{2} \rho_x^{(k)}$$

following from relation (3.21). The above relation gives rise to

$$y_{-x}^{(j)} = -\frac{1}{2} \rho_x^{(k)} + y_-^{(j)} y_-^{(i,j)} = y_-^{(j)} \left( y_-^{(i,j)} - y_+^{(k)} \right) + \alpha_j, \quad (3.27)$$

where use was made of equation (3.21) in the form  $y_-^{(j)} y_+^{(k)} = \frac{1}{2} \rho_x^{(i)}$  and where we introduced

$$\alpha_j = -\frac{1}{2} \rho_x^{(k)} + \frac{1}{2} \rho_x^{(i)} = 2(v_i - v_k).$$

Next consider equation (3.22). It yields :

$$\rho_x^{(k)} = y_{+,x}^{(k)} + y_+^{(k)} \left( -y_+^{(k)} - 2x \right) - 2\nu^{(k)} = y_{+,x}^{(k)} + y_+^{(k)} \left( y_-^{(j)} + y_-^{(i,j)} \right) - 2\nu^{(k)}$$

Rewriting  $\rho_x^{(k)}$  as  $2\alpha_j + \rho_x^{(i)} = -2\alpha_j + 2y_+^{(k)}y_-^{(j)}$  one arrives at

$$y_{+,x}^{(k)} = y_+^{(k)} \left( y_-^{(j)} - y_-^{(i,j)} \right) + 2\nu^{(k)} - 2\alpha_j = y_+^{(k)} \left( y_-^{(j)} - y_-^{(i,j)} \right) + \alpha_k \quad (3.28)$$

where for  $\nu^{(k)} = \nu = -3v_k$  :

$$\alpha_k = 2\nu^{(k)} - 2\alpha_j = -6v_k - 4(v_i - v_k) = 2(v_j - v_i)$$

The final result for the derivative of  $y_-^{(i,j)}$

$$\begin{aligned} y_{-x}^{(i,j)} &= y_-^{(i,j)} \left( y_+^{(k)} - y_-^{(j)} \right) + \alpha_{i,j}, \\ \alpha_{i,j} &= -\alpha_j - \alpha_k - 2 = 2(v_k - v_j) - 2 \end{aligned} \quad (3.29)$$

is obtained by taking a derivative of relation  $y_-^{(i,j)} + y_-^{(j)} + y_+^{(k)} = -2x$ . By setting  $y_+^{(k)} = f_1, y_-^{(j)} = f_2, y_-^{(i,j)} = f_0$  and  $\alpha_k = \alpha_1, \alpha_j = \alpha_2, \alpha_{i,j} = \alpha_0$  we can summarize the above three equations in a form of the symmetric Painlevé IV equation [16, 18]:

$$f_{j,x} = f_j (f_{j+1} - f_{j+2}) + \alpha_j, \quad f_{j+3} = f_j, \quad j = 0, 1, 2 \quad (3.30)$$

Obviously, the association between  $f$ 's and  $y$ 's could have been chosen differently. This does not matter in view of the obvious symmetry for equation (3.30):

$$\pi(f_j) = f_{j+1}, \quad \pi(\alpha_j) = \alpha_{j+1}, \quad (3.31)$$

### 3.3 Bäcklund transformation $\mathbf{G} : \mathbf{y}_- \rightarrow \mathbf{y}_+$

Recall the definition (3.17) of two solutions  $y_{\pm}^{(i)}$  of the Painlevé IV eq. (1.1) with  $b = \nu \pm 1$ . We now define the Bäcklund transformation  $G_k$  such that:

$$G_k(y_-^{(k)}) = y_+^{(k)}. \quad (3.32)$$

$G_k$  takes a solution of the Painlevé IV equation with  $(\mu^{(k)}, \nu^{(k)} - 2)$  to a new solution with  $(\mu^{(k)}, \nu^{(k)})$ . As follows from the definition (3.17) these solutions are related through

$$y_+^{(k)} = y_-^{(k)} - (\ln(\rho_x^{(k)}))_x, \quad k = 1, 2, 3. \quad (3.33)$$

Applying  $G_k$  on both sides of relation (3.33) we get

$$G_k(y_+^{(k)}) = y_+^{(k)} - G(\ln \rho_x^{(k)})_x. \quad (3.34)$$

Applying  $G_k$  on both sides of eq. (3.22) we get  $G(\rho_x^{(k)}) = \rho_x^{(k)} - 2y_{+,x}^{(k)} - 4$ , which after integration yields:

$$G_k(\rho^{(k)}) = \rho^{(k)} + 2 \left( -2x - y_+^{(k)} \right),$$

for all  $k = 1, 2, 3$ . Due to the relation (3.26) we can cast the last equation into

$$G_k(\rho^{(k)}) = \rho^{(k)} + 2 \left( y_-^{(i,j)} + y_-^{(j)} \right) \quad (3.35)$$

Because  $G_k$  increases  $\nu = -3v_k$  by 2 and keeps  $\mu^2 = (v_i - v_j)^2$  invariant we find in accordance with the condition  $\sum_n v_n = 0$  and a choice  $\mu = v_i - v_j$  that

$$G_k(v_i, v_j, v_k) = \left( v_i - \frac{1}{3}, v_j - \frac{1}{3}, v_k + \frac{2}{3} \right) \quad (3.36)$$

for all  $k$  in agreement with identity (1.2) given in section 1. In terms of the identity (3.26) rewritten as

$$y_-^{(k)} + y_+^{(i)} + y_+^{(j,i)} = y_+^{(k)} + y_-^{(i,j)} + y_-^{(j)} = -2x, \quad (3.37)$$

$G_k$  maps term by term the three terms on the left hand side of the above identity to the three terms the right hand side in order of their appearance. Applying to identity (3.37) the permutation operator  $\pi_{ik}$ , which interchanges indices  $i$  and  $k$  without changing the order of terms on both sides, will define  $G_i$  as a map which transforms term by term the left hand side of such new identity to the right hand side. Similarly, applying the permutation operator  $\pi_{jk}$  to identity (3.37) will lead to a new map  $G_j$ .

We will now introduce three Bäcklund transformations  $g_n$  which square to  $G_n$  as given by equations (3.34), (3.35) and (3.36) for the three different values of  $n$  and which generalize relation (2.19).

We first introduce  $g_k$  defined as :

$$g_k \left( y_+^{(i)} \right) = y_-^{(j)}, \quad g_k \left( y_-^{(j)} \right) = y_-^{(i,j)} = y_+^{(i)} + \left( \ln \left( y_-^{(j)} \right) \right)_x \quad (3.38)$$

with

$$g_k(y_+^{(k)}) = y_+^{(k)} - \left( \ln \left( y_-^{(i,j)} \right) \right)_x, \quad (3.39)$$

and

$$g_k(\rho^{(k)}) = \rho^{(k)} + 2y_-^{(j)}, \quad (g_k)^{-1}(\rho^{(k)}) = \rho^{(k)} - 2y_+^{(i)}.$$

The choice of indices  $i, j, k$  in the above definition adheres with the identity

$$y_+^{(k,i)} + y_+^{(i)} + y_-^{(j)} = y_+^{(k)} + y_-^{(j)} + y_-^{(i,j)} = -2x, \quad (3.40)$$

which follows from relation (3.26). Since  $g_k \left( y_-^{(k)} - \left( \ln \left( y_+^{(i)} \right) \right)_x \right) = y_+^{(k)}$  it follows that  $g_k$  keeps the above identity invariant by mapping term by term the left hand side to the right hand side of equation. Thus on basis of arguments of subsection 3.2 this observation shows that  $g_k$  is a Bäcklund transformation of the symmetric Painlevé IV equation. Applying to identity (3.40) the permutation operator  $\pi_{ik}$ , yields a new map  $g_i$  defined explicitly below. Similarly, applying the permutation operator  $\pi_{jk}$  followed by  $\pi_{ik}$  gives  $g_j$  shown explicitly below.

Furthermore for  $g_k$  acting on  $y_+^{(k)}$  and  $\rho^{(k)}$  it holds that  $g_k^2 = G_k$  as shown below step-by-step:

$$\begin{aligned} g_k^2(y_-^{(k)}) &= g_k^2\left(y_+^{(k)} + \left(\ln(\rho_x^{(k)})\right)_x\right) = g_k^2\left(y_+^{(k)} + \left(\ln\left(y_+^{(i)} y_-^{(j)}\right)\right)_x\right) \\ &= g_k\left(y_+^{(k)} + \left(\ln\left(y_-^{(j)}\right)\right)_x\right) = y_+^{(k)} \end{aligned}$$

and

$$\begin{aligned} g_k^2(y_+^{(k)}) &= g_k\left(y_+^{(k)} - \left(\ln\left(y_-^{(i,j)}\right)\right)_x\right) = y_+^{(k)} - \left(\ln\left(g_k^2(\rho_x^{(k)})\right)\right)_x \\ g_k^2(\rho^{(k)}) &= \rho^{(k)} + 2\left(y_-^{(j)} + y_-^{(i,j)}\right) = \rho^{(k)} + 2\left(-2x - y_+^{(k)}\right) \end{aligned}$$

for distinct  $i, j, k$ . The above expressions coincide with the  $(g_k)^2 = G_k$  formula in agreement with (3.32), (3.34) and (3.35). From first of equations (3.38) one sees that  $g_k$  maps  $\nu^{(i)} = -3v_i$  into  $\nu^{(j)} - 2 = -3v_j - 2$ . Thus  $\tilde{v}_i = v_j + \frac{2}{3}$ , where  $\tilde{v}_i$  is a value of parameter  $v_i$  transformed under  $g_k$ . From identity  $(g_k)^2 = G_k$  when applied to  $y_+^{(k)}$  we find that  $g_k$  maps  $\nu^{(k)} = -3v_k$  into  $\nu^{(j)} + 1 = -3v_j + 1$  and thus  $\tilde{v}_k = v_k - \frac{1}{3}$ . In view of relation  $\sum \tilde{v}_n = \sum v_n = 0$  we conclude that

$$g_k(v_i, v_j, v_k) = \left(v_j + \frac{2}{3}, v_i - \frac{1}{3}, v_k - \frac{1}{3}\right), \quad (3.41)$$

which shows that  $\mu^2 = (v_i - v_j)^2 \rightarrow (v_i - v_j - 1)^2$  under  $g_k$ . Acting twice with  $g_k$  yields (in agreement with (3.36))  $G_k(v_i, v_j, v_k) = \left(v_i + \frac{1}{3}, v_j + \frac{1}{3}, v_k - \frac{2}{3}\right)$  for which  $\mu^2 = (v_i - v_j)^2$  remains invariant and  $\nu = -3v_k \rightarrow \nu + 2$ .

Furthermore interchanging  $i \leftrightarrow k$  in the above definition of  $g_k$  yields  $g_i$  defined as :

$$\begin{aligned} g_i\left(y_-^{(j)}\right) &= y_-^{(k,j)} = y_+^{(k)} + \left(\ln\left(y_-^{(j)}\right)\right)_x \\ g_i\left(y_+^{(k)}\right) &= y_-^{(j)} \\ g_i\left(y_+^{(i)}\right) &= y_+^{(i)} - \left(\ln\left(y_-^{(k,j)}\right)\right)_x \\ g_i\left(y_-^{(i,j)}\right) &= y_+^{(i)} = y_-^{(i,j)} - \left(\ln\left(y_-^{(j)}\right)\right)_x \\ g_i\left(\rho^{(i)}\right) &= \rho^{(i)} + 2y_-^{(j)} \end{aligned} \quad (3.42)$$

Therefore

$$\begin{aligned} g_i^2(y_-^{(i)}) &= g_i^2\left(y_+^{(i)} + \left(\ln(\rho_x^{(i)})\right)_x\right) = g_i^2\left(y_+^{(i)} + \left(\ln\left(y_+^{(k)} y_-^{(j)}\right)\right)_x\right) \\ &= g_i\left(y_+^{(i)} + \left(\ln\left(y_-^{(j)}\right)\right)_x\right) = y_+^{(i)} \end{aligned}$$

and

$$\begin{aligned} g_i^2(y_+^{(i)}) &= y_+^{(i)} - \left(\ln\left(g_i^2(y_+^{(k)})\right)\right)_x - \left(\ln g_i^2\left(y_-^{(j)}\right)\right)_x \\ g_i^2(\rho^{(i)}) &= \rho^{(i)} + 2\left(y_-^{(j)} + y_-^{(k,j)}\right) = \rho^{(i)} + 2\left(-2x - y_+^{(i)}\right). \end{aligned}$$

The above agrees with  $g_i^2 = G_i$  when acting on  $y_+^{(i)}, \rho^{(i)}$  variables. Furthermore, from relation (3.41) one obtains through the interchange  $i \leftrightarrow k$  :

$$g_i(v_i, v_j, v_k) = (v_i - \frac{1}{3}, v_k - \frac{1}{3}, v_j + \frac{2}{3}) \quad (3.43)$$

and  $G_i(v_i, v_j, v_k) = (v_i - \frac{2}{3}, v_j + \frac{1}{3}, v_k + \frac{1}{3})$ .

We now propose a square root  $g_j$  of  $G_j$  with respect to its action  $G_j(y_-^{(j)}) = y_+^{(j)}$ . Define, namely

$$\begin{aligned} g_j(y_-^{(j)}) &= y_-^{(j)} - \left( \ln(y_+^{(k)}) \right)_x, & g_j(y_-^{(i,j)}) &= y_+^{(k)} \\ g_j(y_+^{(k)}) &= y_-^{(i,j)} + \left( \ln(y_+^{(k)}) \right)_x & &= y_-^{(i)} \end{aligned} \quad (3.44)$$

Then

$$\begin{aligned} (g_j)^2(y_-^{(j)}) &= g_j \left( y_-^{(j)} - \left( \ln(y_+^{(k)}) \right)_x \right) \\ &= y_-^{(j)} - \left( \ln(y_+^{(k)}) \right)_x - \left( \ln(y_-^{(i,j)} + \left( \ln(y_+^{(k)}) \right)_x) \right)_x \\ &= y_-^{(j)} - \left( \ln(y_+^{(k)}) \right)_x - \left( \ln(y_-^{(i)}) \right)_x = y_-^{(j)} - \left( \ln(y_+^{(k)} y_-^{(i)}) \right)_x \\ &= y_-^{(j)} - \left( \ln(\rho_x^{(j)}) \right)_x = y_+^{(j)} \end{aligned}$$

Use was made of

$$\begin{aligned} y_+^{(j)} &= y_-^{(j)} - \left( \ln(\rho_x^{(j)}) \right)_x \\ y_-^{(i)} &= y_-^{(i,j)} + \left( \ln(y_+^{(k)}) \right)_x \end{aligned}$$

as follows from definition (3.17). Comparing with the transformation rules (3.44) yields

$$g_j(v_i, v_j, v_k) = (v_k - \frac{1}{3}, v_j - \frac{1}{3}, v_i + \frac{2}{3}) \quad (3.45)$$

and  $G_j(v_i, v_j, v_k) = (v_i + \frac{1}{3}, v_j - \frac{2}{3}, v_k + \frac{1}{3})$ .

In a compact notation of the symmetric equation (3.30) based on the association  $y_+^{(k)} = f_1, y_-^{(j)} = f_2, y_-^{(i,j)} = f_0$  we find :

$$\begin{aligned} g_k(f_0) &= f_2 + (\ln f_0)_x, & g_i(f_1) &= f_2, & g_j(f_2) &= f_2 - (\ln f_1)_x \\ g_k(f_1) &= f_1 - (\ln f_0)_x, & g_i(f_2) &= f_1 + (\ln f_2)_x, & g_j(f_0) &= f_1 \\ g_k(f_2) &= f_0, & g_i(f_0) &= f_0 - (\ln f_2)_x, & g_j(f_1) &= f_0 + (\ln f_1)_x. \end{aligned} \quad (3.46)$$

The relevant algebraic relations for the Bäcklund transformations are given in (1.3). In addition it holds that

$$g_k g_i = g_i g_j = g_j g_k = \pi^2, \quad (3.47)$$

where the permutation operator  $\pi$  has been defined in (3.31) and based on the above relations satisfies  $\pi(v_i, v_j, v_k) = (v_k - \frac{1}{3}, v_i - \frac{1}{3}, v_j + \frac{2}{3})$  as is consistent with  $\pi^3 = 1$ .

Relation (3.47) together with relations (1.3) provide realization of the affine Weyl group symmetry in terms of three Bäcklund transformations  $g_n, n = 1, 2, 3$ . Note that the last two identities of relations (1.3) amount to

$$G_j^{-1} = g_k G_j g_k, \quad G_i^{-1} = g_j G_i g_j, \quad G_k^{-1} = g_i G_k g_i.$$

Combining some of the above operations one finds:

$$\pi G_i = G_k \pi, \quad \pi G_k = G_j \pi, \quad \pi G_j = G_i \pi,$$

and

$$g_j = G_i^{-1} \pi_{ik}, \quad g_i = G_j^{-1} \pi_{jk}, \quad g_k = G_j^{-1} \pi_{ij}. \quad (3.48)$$

Equation (3.48) shows that a composition of the DB and the permutation transformations yields the Bäcklund transformations  $g_n, n = 1, 2, 3$ . This observation will come to good use in section 4 as a tool to construct a set of Painlevé IV solutions which closes under the Bäcklund transformations.

We now write down three transformations :

$$s_0 = g_k \pi^2, \quad s_1 = g_j \pi^2, \quad s_2 = g_i \pi^2 \quad (3.49)$$

introduced in [18] (and pedagogically described in [16]) to conveniently interpret the above three maps in terms of root systems associated to the affine Weyl group  $A_2$ .

It now follows from relations (3.46) that these three transformations satisfy

$$\begin{aligned} s_0(f_0) &= f_0, & s_1(f_1) &= f_1, & s_2(f_2) &= f_2 \\ s_0(f_1) &= f_2 + (\ln f_0)_x, & s_1(f_2) &= f_0 + (\ln f_1)_x, & s_2(f_0) &= f_1 + (\ln f_2)_x \\ s_0(f_2) &= f_1 - (\ln f_0)_x, & s_1(f_0) &= f_2 - (\ln f_1)_x, & s_2(f_1) &= f_0 - (\ln f_2)_x \end{aligned} \quad (3.50)$$

and

$$\begin{aligned} s_0(v_i, v_j, v_k) &= (v_i, v_k - 1, v_j + 1) \\ s_1(v_i, v_j, v_k) &= (v_j, v_i, v_k) \\ s_2(v_i, v_j, v_k) &= (v_k, v_j, v_i) \end{aligned} \quad (3.51)$$

One easily finds that  $s_i^2 = 1, i = 0, 1, 2$ .

Based on definition (3.50) one checks that  $s_1(y_-^{(j)}) = y_-^{(i)}, s_1(y_-^{(i)}) = y_-^{(j)}$  and  $s_2(y_+^{(i)}) = y_+^{(k)}, s_2(y_+^{(k)}) = y_+^{(i)}$ . Thus

$$s_1 = \pi_{ij}, \quad s_2 = \pi_{ik} \quad (3.52)$$

as already suggested by relation (3.51).

The following “inverse” relations :

$$g_k = s_0 \pi = \pi s_2, \quad g_i = s_2 \pi = \pi s_1, \quad g_j = s_1 \pi = \pi s_0$$

hold between  $s$ - and  $g$ -Bäcklund transformations.

Let us consider the  $s_1$  and  $s_2$  maps from eq. (3.51) and set for simplicity  $(j, i, k) = (1, 2, 3)$ . Then  $s_1$  and  $s_2$  maps can be geometrically realized as reflections in the hyperplane orthogonal to vectors  $\vec{\alpha}_1 = \vec{e}_1 - \vec{e}_2$  and  $\vec{\alpha}_2 = \vec{e}_2 - \vec{e}_3$ . The  $s_0$  map from eq. (3.51) can then be realized as a reflection in a hyperplane  $\{\vec{x} : \langle \vec{\alpha}_0 | \vec{e}_j - \vec{e}_k \rangle = 1\}$ , with  $\vec{\alpha}_0 = \vec{e}_1 - \vec{e}_3$  being the highest root of  $A_2$  (see f.i. [10]). In terms of  $\alpha_j = \alpha_2 = 2(v_2 - v_3)$ ,  $\alpha_k = \alpha_1 = 2(v_1 - v_2)$  and  $\alpha_{i,j} = \alpha_0 = -\alpha_j - \alpha_k - 2 = 2(v_3 - v_1) - 2$ , introduced in subsection 3.2, the  $s_i$ 's maps can be expressed by the Cartan matrix  $A_{ij}$  [16]:

$$\alpha_j \xrightarrow{s_i} \alpha_j - A_{ij}\alpha_i, \quad i, j = 0, 1, 2$$

Thus we recover from our symmetry structure the extended affine Weyl group structure in the form denoted by Noumi in [16] as  $\widetilde{W} = \langle s_0, s_1, s_2; \pi \rangle$ . It is here introduced only on basis of the three Bäcklund transformations  $g_n, n = 1, 2, 3$ .

### 3.4 From symmetric Painlevé equation to equation for $\rho$

We now take as a starting point the symmetric Painlevé equation (3.30). Our goal is to derive the  $\rho$ -function and its equation (2.8) solely from the quantities entering equation (3.30). Connection from the symmetric Painlevé IV equation to the Hamiltonian formalism was already established in [17]. This subsection provides an alternative proof for that relation.

Out of the three transformation  $g_n, n = 1, 2, 3$  we will chose  $g_j$  defined in eq. (3.44) to illustrate our construction. We denote the action of  $g_j$  by  $\sim$ . Then

$$\begin{aligned} \tilde{f}_2 &= f_2 - (\ln f_1)_x = f_0 - \frac{\alpha_1}{f_1}, & \tilde{\alpha}_2 &= -2 - \alpha_2 \\ \tilde{f}_0 &= f_1, & \tilde{\alpha}_0 &= -\alpha_1 \\ \tilde{f}_1 &= f_0 + (\ln f_1)_x = f_2 + \frac{\alpha_1}{f_1}, & \tilde{\alpha}_1 &= \alpha_1 + \alpha_2 \end{aligned} \quad (3.53)$$

For the transformed functions it still holds that  $\tilde{f}_0 + \tilde{f}_1 + \tilde{f}_2 = -2x$ . Also they satisfy the symmetric Painlevé eq. (3.30) with the coefficients  $\tilde{\alpha}_i, i = 0, 1, 2$  given above.

Acting twice with this transformation yields

$$\begin{aligned} \bar{f}_0 &= \tilde{\tilde{f}}_0 = \tilde{f}_1 = f_2 + \frac{\alpha_1}{f_1} & \bar{\alpha}_0 &= -\alpha_1 - \alpha_2 \\ \bar{f}_2 &= \tilde{\tilde{f}}_2 = f_1 \frac{f_1 f_2 - \alpha_2}{f_1 f_2 + \alpha_1} & \bar{\alpha}_2 &= \alpha_2, \\ \bar{f}_1 &= \tilde{\tilde{f}}_1 = f_0 - \frac{\alpha_1}{f_1} + \frac{\alpha_1 + \alpha_2}{f_2 + \alpha_1/f_1} & \bar{\alpha}_1 &= -2 + \alpha_1 \end{aligned} \quad (3.54)$$

which describes an action of  $G_j$ . Because  $G_j$  operation is a symmetry it follows that  $\bar{f}_i, i = 0, 1, 2$  satisfy as well the symmetric Painlevé eq. (3.30) with coefficients  $\bar{\alpha}_i, i = 0, 1, 2$  given above. Next, define the quantities :

$$\begin{aligned} \frac{1}{2}\sigma_x^{(j)} &= f_0 f_1 + f_{1x} = f_1 f_2 + \alpha_1, & \frac{1}{2}\sigma_x^{(i)} &= f_1 f_2, \\ \frac{1}{2}\sigma_x^{(k)} &= f_0 f_2 - f_{2x} = f_1 f_2 - \alpha_2 \end{aligned} \quad (3.55)$$

It follows that

$$\bar{f}_2 - f_2 = -(\ln(f_1 f_2 + \alpha_1))_x = -(\ln \sigma_x^{(j)})_x \quad (3.56)$$

Also the following important Riccati equation

$$\begin{aligned} -f_{2,x} - f_2^2 - 2x f_2 &= -f_2(f_0 - f_1) - \alpha_2 - f_2^2 + (f_0 + f_1 + f_2)f_2 \\ &= 2f_1 f_2 - \alpha_2 = \sigma_x^{(j)} - 2\alpha_1 - \alpha_2 \end{aligned} \quad (3.57)$$

follows from the symmetric Painlevé eq. (3.30).

We want to show that the transformed version of  $\sigma_x^{(j)}$  given by

$$\frac{1}{2}\bar{\sigma}_x^{(j)} = \bar{f}_0 \bar{f}_1 + \bar{f}_{1x}$$

is equal to

$$\frac{1}{2}\bar{\sigma}_x^{(j)} = \frac{1}{2}\sigma_x^{(j)} - 2 - \bar{f}_{2x}$$

We accomplish this in two steps. First we apply the  $\tilde{\cdot}$  transformation on relation (3.55) which yields

$$\frac{1}{2}\tilde{\sigma}_x^{(j)} = \tilde{f}_0 \tilde{f}_1 + \tilde{f}_{1x} = f_1(f_0 + (\ln f_1)_x) + \tilde{f}_{1x} = f_0 f_1 + f_{1,x} + \tilde{f}_{1x} = \frac{1}{2}\sigma_x^{(j)} + \tilde{f}_{1x}$$

Applying another  $\tilde{\cdot}$  transformation on the above equation yields

$$\begin{aligned} \frac{1}{2}\bar{\sigma}_x^{(j)} &= \frac{1}{2}\tilde{\sigma}_x^{(j)} + \bar{f}_{1x} = \frac{1}{2}\sigma_x^{(j)} + \tilde{f}_{1,x} + \bar{f}_{1x} \\ &= \frac{1}{2}\sigma_x^{(j)} + \bar{f}_2(\bar{f}_1 - \bar{f}_0) - 2 - \alpha_2 = \frac{1}{2}\sigma_x^{(j)} - 2 - \bar{f}_{2x} \end{aligned}$$

where we used :  $\tilde{f}_0 = \bar{f}_2 + \bar{\alpha}_1/\bar{f}_0$ ,  $\tilde{f}_1 = \bar{f}_0$  and  $\tilde{f}_2 = \bar{f}_1 - \bar{\alpha}_1/\bar{f}_0$  and  $\bar{\alpha}_1 - \tilde{\alpha}_1 = -2 - \alpha_2$ .

Repeating steps taken in the derivation of the Riccati equation (3.57) and using the result we have established above we arrive at

$$- \bar{f}_{2,x} - \bar{f}_2^2 - 2x \bar{f}_2 = \bar{\sigma}_x^{(j)} - 2\bar{\alpha}_1 - \bar{\alpha}_2 = \sigma_x^{(j)} - 2\bar{f}_{2x} - 2\alpha_1 - \alpha_2$$

Thus we have established a Darboux dressing chain relation for  $f_2$  and  $\bar{f}_2$  :

$$-f_{2,x} - f_2^2 - 2x f_2 = \bar{f}_{2,x} - \bar{f}_2^2 - 2x \bar{f}_2 = \sigma_x^{(j)} - 2\alpha_1 - \alpha_2. \quad (3.58)$$

Plugging on the right hand side of the above equation the expression (3.56) for  $\bar{f}_2$  in terms of  $f_2$  and subtracting from the left hand side yields

$$f_{2,x} + f_2 (\ln \sigma_x^{(j)})_x = \frac{1}{2} (\ln \sigma_x^{(j)})_{xx} + \frac{1}{2} (\ln \sigma_x^{(j)})_x^2 - x (\ln \sigma_x^{(j)})_x \quad (3.59)$$

This is a first order differential equation for which we easily find a particular solution

$$f_2 = \frac{-1}{2\sigma_x^{(j)}} (2(x\sigma_x^{(j)} - \sigma_x^{(j)}) - \sigma_{xx}^{(j)}) \quad (3.60)$$

Note that adding to the above a homogeneous solution of equation (3.59) amounts to adding an integration constant to  $\sigma^{(j)}$ .

Furthermore from the relation (3.56) one gets

$$\bar{f}_2 = \frac{-1}{2\sigma_x^{(j)}} (2(x\sigma_x^{(j)} - \sigma^{(j)}) + \sigma_{xx}^{(j)}) \quad (3.61)$$

The last two equations agree with relation (3.11) for  $\sigma^{(j)} = \rho^{(j)}$ ,  $f_2 = y_-^{(j)}$  and  $\bar{f}_2 = y_+^{(j)}$ .

Recall from (3.54) that

$$\bar{f}_2 = f_1 \frac{f_1 f_2 - \alpha_2}{f_1 f_2 + \alpha_1}$$

from the above relation and definition (3.55) we conclude that

$$f_2 \bar{f}_2 = f_1 f_2 \frac{f_1 f_2 - \alpha_2}{f_1 f_2 + \alpha_1} = \frac{1}{2} \sigma_x^{(i)} \frac{1}{2} \sigma_x^{(k)} \left( \frac{1}{2} \sigma_x^{(j)} \right)^{-1} \quad (3.62)$$

Plugging into the above expressions (3.60) and (3.61) we recover

$$(2(x\sigma_x^{(j)} - \sigma^{(j)}) - \sigma_{xx}^{(j)}) (2(x\sigma_x^{(j)} - \sigma^{(j)}) + \sigma_{xx}^{(j)}) = 2\sigma_x^{(i)} \sigma_x^{(j)} \sigma_x^{(k)} \quad (3.63)$$

which shows that  $\sigma^{(j)}$  satisfies the  $\rho$ -equation (2.8). Since the left hand side of equation (3.63) is unaffected by a shift of  $\sigma_x$  by a constant the relation also holds for  $\sigma^{(i)}$  and  $\sigma^{(k)}$ . The same conclusion also follows from the same consideration as given above when applied to  $g_i$  and  $g_k$ .

## 4 Applications

### 4.1 the “ $-2x$ and $-1/x$ ”-hierarchies

Recall a basic solution of the “ $-2x$  hierarchy” [6, 7, 12, 18]

$$\rho_{(k,n)} = 2\partial_x \ln W_k[H_n, H_{n-1}, \dots, H_{n-k+1}], \quad k > 1, \quad (4.1)$$

which is directly obtained as a reduction of the soliton solutions of the AKNS hierarchy as discussed in [2]. Here,  $H_n(x)$  is the Hermite polynomial and  $\rho_{(k,n)}$  satisfies  $\rho$ -equation (2.8) with  $\nu = n - 2k + 1$  and  $\mu^2 = (n+1)^2$ . Formula (2.9) (with a minus subscript) yields:

$$y_{-, (k,n)} = \partial_x \ln \frac{W_{k+1}[H_n, H_{n-1}, \dots, H_{n-k}]}{W_k[H_n, H_{n-1}, \dots, H_{n-k+1}]} - 2x, \quad (4.2)$$

which solves the Painlevé IV equation with the same values of  $\nu$  and  $\mu$  as the solution in (4.1).

Next, we implement  $\rho \rightarrow \rho^{(i)}, i = 1, 2$  symmetry to derive from the above other solutions. We proceed as in (3.12)-(3.13) with  $\mu = (n + 1)$  and  $\nu = n - 2 \cdot k + 1$  :

$$\rho_{(k,n)}^{(1)} = \rho_{(k,n)} - 2 \cdot 2kx, \quad \nu^{(1)} = n + k + 1, \quad \mu^{(1)} = \pm(n - k + 1) \quad (4.3)$$

$$\rho_{(k,n)}^{(2)} = \rho_{(k,n)} + 2 \cdot 2(n - k + 1)x, \quad \nu^{(2)} = -2n + k - 2, \quad \mu^{(2)} = \pm k \quad (4.4)$$

Thus

$$\rho_{(k,n)}^{(1)} = 2\partial_x \ln W_k[e^{-x^2} H_n, e^{-x^2} H_{n-1}, \dots, e^{-x^2} H_{n-k+1}], \quad (4.5)$$

which via formula (2.9)(with the “+” subscript) gives rise to

$$y_{(k,n)}^{(1)} = \partial_x \ln \frac{W_k[H_n, H_{n-1}, \dots, H_{n-k+1}]}{W_{k+1}[H_{n+1}, H_n, \dots, H_{n-k+1}]} = \partial_x \ln \frac{W_{n-k+1}[\widehat{H}_n, \widehat{H}_{n-1}, \dots, \widehat{H}_k]}{W_{n-k+1}[\widehat{H}_{n+1}, \widehat{H}_n, \dots, \widehat{H}_{k+1}]}, \quad (4.6)$$

where  $\widehat{H}_m(x) = e^{-x^2} d^m e^{x^2} / dx^m = (-i)^m H_m(ix)$ . The function in eq. (4.6) solves the Painlevé IV equation with  $\nu = n + k + 1$  and  $\mu^2 = (n - k + 1)^2$  and belongs to the so-called “ $-1/x$ -hierarchy” but here is obtained as a result of the simple symmetry transformation  $\rho \rightarrow \rho^{(1)}$ .

A convenient expression for  $\rho_{(k,n)}^{(2)}$  from equation (4.4) is

$$\rho_{(k,n)}^{(2)} = 2\partial_x \ln W_{n-k+1}[e^{x^2} \widehat{H}_n, e^{x^2} \widehat{H}_{n-1}, \dots, e^{x^2} \widehat{H}_k] \quad (4.7)$$

and via formula (2.9) (with “+” subscript) it gives rise to

$$y_{(k,n)}^{(2)} = \partial_x \ln \frac{W_{n-k+1}[\widehat{H}_n, \widehat{H}_{n-1}, \dots, \widehat{H}_k]}{W_{n-k}[\widehat{H}_{n-1}, \dots, \widehat{H}_k]} = \partial_x \ln \frac{W_k[H_n, H_{n-1}, \dots, H_{n-k+1}]}{W_k[H_{n-1}, H_{n-2}, \dots, H_{n-k}]} \quad (4.8)$$

which solves the Painlevé IV equation with  $\nu = -2n + k - 2$  and  $\mu^2 = k^2$ . Substituting  $k' = n - k, n' = n - 1$  we can rewrite the above as:

$$y_{(k,n)}^{(2)} = \partial_x \ln \frac{W_{k'+1}[\widehat{H}_{n'+1}, \widehat{H}_{n'}, \dots, \widehat{H}_{n'-k'+1}]}{W_{k'}[\widehat{H}_{n'}, \widehat{H}_{n'-1}, \dots, \widehat{H}_{n'-k'+1}]} = \partial_x \ln \frac{W_{n'-k'+1}[H_{n'+1}, H_{n'}, \dots, H_{k'+1}]}{W_{n'-k'+1}[H_{n'}, H_{n'-1}, \dots, H_{k'}]} \quad (4.9)$$

Also  $y_{(k,n)}^{(2)}$  denoted as  $w^{(k',n')}$  belongs to the so-called “ $-1/x$ -hierarchy” with the parameters  $\nu = -n' - k' - 3, \mu = \pm(n' - k' + 1)$ .

The following version of identity (3.26) holds here :

$$y_{-, (k,n)} + y_{(k,n)}^{(1)} + y_{(k,n)}^{(2)} = -2x - (\ln y_{-, (k,n)})_x \quad (4.10)$$

Next, recall (c.f. [2]) the remaining basic solution of the “ $-2x$  hierarchy”

$$\widehat{\rho}^{(k,n)} = 2\partial_x \ln W_k[\widehat{H}_n, \widehat{H}_{n-1}, \dots, \widehat{H}_{n-k+1}]$$

which satisfies  $\rho$ -equation (2.8) with  $\nu = -n + 2 \cdot k - 1$  and  $\mu^2 = (n + 1)^2$ . Using relation (2.9) (with a “minus” subscript) yields the corresponding solution to the Painlevé IV equation :

$$\widehat{y}_{-}^{(k,n)} = -\partial_x \ln \frac{W_k[\widehat{H}_n, \widehat{H}_{n-1}, \dots, \widehat{H}_{n-k+1}]}{W_{k-1}[\widehat{H}_n, \widehat{H}_{n-1}, \dots, \widehat{H}_{n-k+2}]} - 2x \quad (4.11)$$

with  $b = -n + 2k - 2 = \nu - 1$  and  $\mu^2 = (n + 1)^2$ .

We proceed as in (3.12)-(3.13) with  $\mu = (n + 1)$  to obtain :

$$\begin{aligned}\hat{\rho}_{(k,n)}^{(1)} &= \hat{\rho}_{(k,n)} - 2 \cdot 2(n - k + 1)x, \quad \nu^{(1)} = 2n - k + 2, \quad \mu^{(1)} = \pm k \\ &= 2\partial_x \ln W_{n-k+1}[e^{-x^2}H_n, e^{-x^2}H_{n-1}, \dots, e^{-x^2}H_k]\end{aligned}\quad (4.12)$$

$$\begin{aligned}\hat{\rho}_{(k,n)}^{(2)} &= \hat{\rho}_{(k,n)} + 2 \cdot 2kx, \quad \nu^{(2)} = -n - k - 1, \quad \mu^{(2)} = \pm(n - k + 1) \\ &= 2\partial_x \ln W_k[e^{x^2}\hat{H}_n, e^{x^2}\hat{H}_{n-1}, \dots, e^{x^2}\hat{H}_{n-k+1}]\end{aligned}\quad (4.13)$$

Via formula (2.9) they give rise to

$$\hat{y}_{(k,n)}^{(1)} = \partial_x \ln \frac{W_{n-k+2}[H_{n+1}, H_{n-1}, \dots, H_k]}{W_{n-k+1}[\hat{H}_n, \dots, H_k]} = \partial_x \ln \frac{W_k[\hat{H}_{n+1}, \hat{H}_{n-1}, \dots, \hat{H}_{n-k+2}]}{W_k[\hat{H}_n, \hat{H}_{n-2}, \dots, \hat{H}_{n-k+1}]}\quad (4.14)$$

which solves the Painlevé IV equation with  $\nu = 2n - k + 2$  and  $\mu^2 = k^2$  and

$$\hat{y}_{(k,n)}^{(2)} = \partial_x \ln \frac{W_k[\hat{H}_n, \hat{H}_{n-1}, \dots, \hat{H}_{n-k+1}]}{W_{k-1}[\hat{H}_{n-1}, \dots, \hat{H}_{n-k+1}]} = \partial_x \ln \frac{W_{n-k+1}[H_n, H_{n-1}, \dots, H_k]}{W_{n-k+1}[H_{n-1}, H_{n-2}, \dots, H_{k-1}]}\quad (4.15)$$

with  $\nu^{(2)} = -n - k - 1$ ,  $\mu^{(2)} = \pm(n - k + 1)$ .

It holds that

$$\hat{y}_{-}^{(k,n)} + \hat{y}_{(k,n)}^{(1)} + \hat{y}_{(k,n)}^{(2)} = -2x - \left( \ln \hat{y}_{-}^{(k,n)} \right)_x. \quad (4.16)$$

## 4.2 “ $-2x/3$ -hierarchy”

We will introduce the following polynomials :

$$F_n^{(k)} = \frac{e^{x^2/3}}{2^n n!} \frac{d^{3n+k}}{dx^{3n+k}} e^{-x^2/3}, \quad \hat{F}_n^{(k)} = \frac{e^{-x^2/3}}{2^n n!} \frac{d^{3n+k}}{dx^{3n+k}} e^{x^2/3}, \quad (4.17)$$

defined for  $k = 0, 1, 2$ ,  $n = 0, 1, 2, \dots$

A starting point is a basic polynomial solution of equation (2.8):

$$\rho^{(0)} = \frac{8}{27}x^3, \quad (\mu^{(0)})^2 = \frac{4}{9}, \quad \nu^{(0)} = 0. \quad (4.18)$$

We proceed as in relations (3.12)-(3.13) with  $\mu^{(0)} = 2/3$  to obtain :

$$\rho^{(1)} = \frac{8}{27}x^3 - \frac{4}{3}x, \quad (\mu^{(1)})^2 = \frac{1}{9}, \quad \nu^{(1)} = +1 \quad (4.19)$$

$$\rho^{(2)} = \frac{8}{27}x^3 + \frac{4}{3}x, \quad (\mu^{(2)})^2 = \frac{1}{9}, \quad \nu^{(2)} = -1. \quad (4.20)$$

Then repeated actions with  $G^{(\pm 1)}$  on  $\rho^{(0)}$  yield:

$$\begin{aligned}
\rho^{(0,n)} &= G^n(\rho^{(0)}) = \rho^{(0)} - 2n \frac{4x}{3} + 2 \left( \ln W_n[F_0^{(1)}, F_1^{(1)}, \dots, F_{n-1}^{(1)}] \right)_x \\
&= \rho^{(0)} + 2 \left( \ln W_n[e^{\frac{-2x^2}{3}} F_0^{(1)}, e^{\frac{-2x^2}{3}} F_1^{(1)}, \dots, e^{\frac{-2x^2}{3}} F_{n-1}^{(1)}] \right)_x \quad \nu = 2n \\
\rho^{(0,-n)} &= G^{-n}(\rho^{(0)}) = \rho^{(0)} + 2n \frac{4x}{3} + 2 \left( \ln W_n[\widehat{F}_0^{(1)}, \widehat{F}_1^{(1)}, \dots, \widehat{F}_{n-1}^{(1)}] \right)_x \\
&= \rho^{(0)} + 2 \left( \ln W_n[e^{\frac{2x^2}{3}} \widehat{F}_0^{(1)}, e^{\frac{2x^2}{3}} \widehat{F}_1^{(1)}, \dots, e^{\frac{2x^2}{3}} \widehat{F}_{n-1}^{(1)}] \right)_x \quad \nu = -2n,
\end{aligned} \tag{4.21}$$

which are solutions of the  $\rho$ -equation (2.8) with  $\mu^2 = \frac{4}{9}$  and  $\nu = \pm 2n, n = 1, 2, 3, \dots$ , respectively.

Next, proceeding as in (3.12)-(3.13) with respect to  $\rho^{(0,n)}$  yields from (4.21) :

$$\begin{aligned}
\rho^{(1,n)} &= \rho^{(1)} + n \frac{4x}{3} + 2 \left( \ln W_n[F_0^{(1)}, F_1^{(1)}, \dots, F_{n-1}^{(1)}] \right)_x, \\
\mu^{(1,n)} &= \pm \left( \frac{1}{3} + n \right), \quad \nu^{(1,n)} = 1 - n
\end{aligned} \tag{4.22}$$

$$\begin{aligned}
\rho^{(2,n)} &= \rho^{(2)} + n \frac{4x}{3} + 2 \left( \ln W_n[F_0^{(1)}, F_1^{(1)}, \dots, F_{n-1}^{(1)}] \right)_x, \\
\mu^{(2,n)} &= \pm \left( \frac{1}{3} - n \right), \quad \nu^{(2,n)} = -1 - n.
\end{aligned} \tag{4.23}$$

Acting repeatedly with  $G$  according to

$$G(\rho^{(i,n)}) = \rho^{(i,n)} + 2 \left( -2x - y_+^{(i,n)} \right), \quad i = 1, 2$$

yields after  $k$ -times:

$$\begin{aligned}
\rho^{(1,n,k)} &= \rho^{(1)} + (n - 2k) \frac{4x}{3} + 2 \left( \ln W_n[F_0^{(1)}, F_1^{(1)}, \dots, F_{n-1}^{(1)}, F_0^{(2)}, F_1^{(2)}, \dots, F_{k-1}^{(2)}] \right)_x, \\
\mu^{(1,n,k)} &= \pm \left( \frac{1}{3} + n \right), \quad \nu^{(1,n,k)} = 1 - n + 2k
\end{aligned} \tag{4.24}$$

$$\begin{aligned}
\rho^{(2,n,k)} &= \rho^{(2)} + (n - 2k) \frac{4x}{3} + 2 \left( \ln W_n[F_0^{(1)}, F_1^{(1)}, \dots, F_{n-1}^{(1)}, F_0^{(0)}, F_1^{(0)}, \dots, F_{k-1}^{(0)}] \right)_x, \\
\mu^{(2,n,k)} &= \pm \left( \frac{1}{3} - n \right), \quad \nu^{(2,n,k)} = -1 - n + 2k.
\end{aligned} \tag{4.25}$$

The same steps, meaning, first applying (3.12)-(3.13) with respect to  $\rho^{(0,-n)}$  from (4.21) in order to derive  $\rho^{(i,-n)}, i = 1, 2$  and then applying  $G^{-1}$  gives :

$$\begin{aligned}
\rho^{(1,-n,-k)} &= \rho^{(1)} - (n - 2k) \frac{4x}{3} + 2 \left( \ln W_n[\widehat{F}_0^{(1)}, \widehat{F}_1^{(1)}, \dots, \widehat{F}_{n-1}^{(1)}, \widehat{F}_0^{(0)}, \widehat{F}_1^{(0)}, \dots, \widehat{F}_{k-1}^{(0)}] \right)_x, \\
\mu^{(1,-n,-k)} &= \pm \left( \frac{1}{3} - n \right), \quad \nu^{(1,-n,-k)} = 1 + n - 2k
\end{aligned} \tag{4.26}$$

$$\begin{aligned}
\rho^{(2,-n,-k)} &= \rho^{(2)} - (n - 2k) \frac{4x}{3} + 2 \left( \ln W_n[\widehat{F}_0^{(1)}, \widehat{F}_1^{(1)}, \dots, \widehat{F}_{n-1}^{(1)}, \widehat{F}_0^{(2)}, \widehat{F}_1^{(2)}, \dots, \widehat{F}_{k-1}^{(2)}] \right)_x, \\
\mu^{(2,-n,-k)} &= \pm \left( \frac{1}{3} + n \right), \quad \nu^{(2,-n,-k)} = -1 + n - 2k.
\end{aligned} \tag{4.27}$$

One finds by comparing  $\mu$  and  $\nu$ 's of the above expressions that

$$\rho^{(2,n,k)} = \rho^{(1,-n,-(1+n-k))}, \quad n, k = 0, 1, 2, 3, \dots,$$

$$\rho^{(1,n,k)} = \rho^{(2,-n,-(n-1-k))}, \quad n, k = 0, 1, 2, 3, \dots,$$

Equation (3.17) is providing a general construction for a map  $\rho^{(i,n,k)} \rightarrow y_+^{(i,n,k)}$  and  $\rho^{(i,-n,-k)} \rightarrow y_-^{(i,-n,-k)}$  for  $i = 1, 2$  with the following results:

$$y_+^{(1,n,k)} = - \left( \ln \left( \frac{W_{k+n+1}[F_0^{(1)}, F_1^{(1)}, \dots, F_{n-1}^{(1)}, F_0^{(2)}, F_1^{(2)}, \dots, F_k^{(2)}]}{W_{k+n}[F_0^{(1)}, F_1^{(1)}, \dots, F_{n-1}^{(1)}, F_0^{(2)}, F_1^{(2)}, \dots, F_{k-1}^{(2)}]} \right) \right)_x - \frac{2x}{3}, \quad (4.28)$$

$$\mu^2 = \left( \frac{1}{3} + n \right)^2, \quad \nu = 2k - n + 1$$

$$y_+^{(2,n,k)} = - \left( \ln \left( \frac{W_{k+n+1}[F_0^{(1)}, F_1^{(1)}, \dots, F_{n-1}^{(1)}, F_0^{(0)}, F_1^{(0)}, \dots, F_k^{(0)}]}{W_{k+n}[F_0^{(1)}, F_1^{(1)}, \dots, F_{n-1}^{(1)}, F_0^{(0)}, F_1^{(0)}, \dots, F_{k-1}^{(0)}]} \right) \right)_x - \frac{2x}{3}, \quad (4.29)$$

$$\mu^2 = \left( -\frac{1}{3} + n \right)^2, \quad \nu = 2k - n - 1$$

$$y_-^{(1,-n,-k)} = \left( \ln \left( \frac{W_{k+n+1}[\widehat{F}_0^{(1)}, \widehat{F}_1^{(1)}, \dots, \widehat{F}_{n-1}^{(1)}, \widehat{F}_0^{(0)}, \widehat{F}_1^{(0)}, \dots, \widehat{F}_k^{(0)}]}{W_{k+n}[\widehat{F}_0^{(1)}, \widehat{F}_1^{(1)}, \dots, \widehat{F}_{n-1}^{(1)}, \widehat{F}_0^{(0)}, \widehat{F}_1^{(0)}, \dots, \widehat{F}_{k-1}^{(0)}]} \right) \right)_x - \frac{2x}{3}, \quad (4.30)$$

$$\mu^2 = \left( -\frac{1}{3} + n \right)^2, \quad \nu = -2k + n + 1$$

$$y_-^{(2,-n,-k)} = \left( \ln \left( \frac{W_{k+n+1}[\widehat{F}_0^{(1)}, \widehat{F}_1^{(1)}, \dots, \widehat{F}_{n-1}^{(1)}, \widehat{F}_0^{(2)}, \widehat{F}_1^{(2)}, \dots, \widehat{F}_k^{(2)}]}{W_{k+n}[\widehat{F}_0^{(1)}, \widehat{F}_1^{(1)}, \dots, \widehat{F}_{n-1}^{(1)}, \widehat{F}_0^{(2)}, \widehat{F}_1^{(2)}, \dots, \widehat{F}_{k-1}^{(2)}]} \right) \right)_x - \frac{2x}{3}, \quad (4.31)$$

$$\mu^2 = \left( \frac{1}{3} + n \right)^2, \quad \nu = -2k + n - 1$$

The corresponding symmetric form of the Painlevé equation for the above solutions is e.g.

$$y_+^{(1,n,k)} + y_+^{(2,k+1,n+1)} + y_-^{(1,-(n-k),-(n+1))} = -2x - \left( \ln y_-^{(1,-(n-k),-(n+1))} \right)_x, \quad n \geq k$$

and

$$y_+^{(1,n,k)} + y_+^{(2,k+1,n+1)} + y_-^{(2,-(k-n),-k)} = -2x - \left( \ln y_-^{(2,-(k-n),-k)} \right)_x, \quad k \geq n.$$

In view of equation (3.48) the above construction realized the Bäcklund transformations as compositions of the DB and the permutation transformations. Indeed, assigning  $(v_i, v_j, v_k) = ((n+k+1)/3, (k-2n)/3, (n-2k-1)/3)$  to the  $y_{(k,n)}^{(1)}$  solution one verifies that the class of solutions shown in this subsection closes under

the action of the Bäcklund transformations  $g_i, g_j, g_k$ . For example, the transformation  $g_j$  maps  $y_{(k,n)}^{(1)}$  to  $y_{(k',n')}^{(2)}$  with  $n' = n - k, k' = n$  for  $n > k$  and  $\hat{y}_{(-k',-n')}^{(2)}$  with  $n' = k - n, k' = k + 1$  for  $n < k$ .

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